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0. ABSTRACT

Multidimensional Wilson-Hilferty [32] transformations support Gaussian approximations to certain joint distributions of quadratic forms in Gaussian variables. Central and noncentral distributions are studied and applications are noted. Parameters of the approximating distributions are given up to terms of order $O(\nu^{-3})$ in the degrees of freedom ν. Numerical studies validate using these approximations over a range of parameters for an essential subclass of the distributions.

1. INTRODUCTION

Developments in statistics and applied probability often entail joint distributions of definite quadratic forms in Gaussian variables, either in small samples or asymptotically. Examples include linear statistical models [10], the ballistics of multiple weapons systems [4], signal detection in multichannel receivers [25], and bone lengths determined *in vivo* using X-ray stereography ([26], [28] and [29]). Topics in large-sample theory include limiting joint distributions of likelihood ratios, of Pearson's [27] X^2 statistics for categorical data [12], and of Friedman's [5] χ_r^2 statistics in two-way data without normality ([13] and [14]). Further details are given subsequently, and other examples could be cited.

Distributions of these types include multidimensional chi-squared and Rayleigh distributions ([8], [9], [10], [18], [19], [20], [22], [24], and [30]), often depending on an excess of parameters. Series expansions for their distribution functions typically are intractable in dimensions greater than two; convergence properties of these series may not be known [19]; and series known to converge may do so slowly. Viable approximations to these distributions are clearly needed. Among alternatives, multivariate Edgeworth expansions [3] are often flawed by inadequacy of the leading term and failure of the sum of the first few terms to be positive. Approximations from the Pearson system, in wide use in the univariate case, hold scant promise since little is known beyond the bivariate case ([18], *pp.* 6-9). Alternatively, the normalizing transformations of Wilson and Hilferty [32] appear promising, as these give remarkable accuracy for a single quadratic form over a wide range of parameters ([16], [17] and [23]).

Here we develop multivariate transformations leading to Gaussian approximations not depending on excessive parameters, requiring only moments of first and second orders. To be precise, let $[Q_1, \dots, Q_r]$ be standardized quadratic forms in Gaussian variables having ν degrees of freedom whose limit as $\nu \to \infty$ is jointly Gaussian. Lemma 1 assures that $[Q_1^{\alpha_1}, \dots, Q_r^{\alpha_r}]$ also has a Gaussian limit for every $\{\alpha_j \in (0,1];\ 1 \le j \le r\}$. Extending developments in [16], we consider multivariate power transformations $\{Q_j \to (Q_j)^{\alpha_j};\ 1 \le j \le r\}$, choosing $\{\alpha_1, \dots, \alpha_r\}$ so as to accelerate the convergence of certain moment sequences to those of Gaussian variables. Parameters of the approximating distributions are given up to terms of order $o(\nu^{-3})$, and the approximation is studied numerically in selected cases.

2. PRELIMINARIES

2.1 Notation. Standard spaces include the Euclidean n-space $\mathbb{R}^n$, the space $F_{n\times p}$ of real $(n \times p)$ matrices, the symmetric $(p \times p)$ matrices S_p, and the cone $S_p^+ \subset S_p$ of positive semidefinite varieties. The half-open unit cube in $\mathbb{R}^n$ is denoted by $D(n) = (0, 1] \times (0, 1] \times \dots \times (0, 1]$. Matrix operations include transposition $(A \to A')$ and inversion $(B \to B^{-1})$. Special arrays are I_n, 0, $A \times B$, and $\mathrm{Diag}(A_1, \dots, A_r)$, denoting respectively the $(n \times n)$ identity, a matrix of zeros, the direct product $[a_{ij}B]$, and a block-diagonal matrix, in addition to the unit vector $1_n = [1, \dots, 1]' \in \mathbb{R}^n$.

2.2 Some Basic Distributions. $\mathscr{L}(Z)$ denotes the law of distribution of Z. The joint moments of $X \in \mathbb{R}^p$ are denoted by $\mu'_{s_1 \dots s_p}$, its central moments by $\mu_{s_1 \dots s_p}$, and its joint cumulants by $\kappa_{s_1 \dots s_p}$, all of order $s = s_1 + \cdots + s_p$. We consider the probability density function (*pdf*), the cumulative distribution function (*cdf*), and the cumulant generating function (*cgf*) of Z. Gaussian laws on $\mathbb{R}^p$ and $F_{n\times p}$ are denoted by $N_p(\mu, \Sigma)$ and $N_{n\times p}(M, \Gamma)$, respectively, where $\mu \in \mathbb{R}^p$ and $M \in F_{n\times p}$ are arrays of means, and $\Sigma \in S_p^+$ and $\Gamma \in S_{np}^+$ are arrays of dispersion parameters. The Wishart distribution on S_p^+, having ν degrees of freedom and the matrices Σ and Θ of scale and noncentrality parameters in S_p^+, is denoted by $W_p(\nu, \Sigma, \Theta)$. In particular, if $\mathscr{L}(Y) = N_{n\times p}(M, I_n \times \Sigma)$, then $\mathscr{L}(Y'Y) = W_p(n, \Sigma, M'M)$.

2.3 Quadratic Forms. We represent quadratic forms in Gaussian variables *via* partitioned Wishart matrices. Suppose that $\mathscr{L}(y) = N_p(\mu, \Sigma)$; consider quadratic forms $[y'A_1y, \dots, y'A_ry]$ such that $\{A_j \in S_p^+;\ 1 \le j \le r\}$; identify the joint distribution of $z = [z_1', \dots, z_r']'$ on $\mathbb{R}^{rp}$ as $N_{rp}(\theta, \Gamma)$, where $\{z_j = A_j^{1/2}y;\ 1 \le j \le r\}$, $\theta = [\theta_1', \dots, \theta_r']'$ and $\{\theta_j = A_j^{1/2}\mu;\ 1 \le j \le r\}$; and write $\Gamma \in S_{rp}^+$ in partitioned form as $\Gamma = [A_i^{1/2}\Sigma A_j^{1/2}]$. If

$W = zz'$, then $\mathscr{L}(W) = W_{rp}(1, \Gamma, \theta\theta')$. Partitioning $W = [W_{ij}]$ conformably with $[z_1', \dots, z_r']'$ shows that $[y'A_1y, \dots, y'A_ry]$ may be represented equivalently as $[z_1'z_1, \dots, z_r'z_r]$ and $[\mathrm{tr}W_{11}, \dots, \mathrm{tr}W_{rr}]$.

Generally $[\mathrm{tr}W_{11}, \dots, \mathrm{tr}W_{rr}]$ are quadratic forms in jointly Gaussian variables. For if $W = Y'Y$ with $\mathscr{L}(Y) = N_{n\times p}(M, I_n \times \Sigma)$, then $\mathscr{L}(W) = W_p(n, \Sigma, \Theta)$ with $\Theta = M'M$. Partitioning the typical row of Y as $y_i' = [y_{i1}', \dots, y_{ir}']$ such that $y_{ij} \in \mathbb{R}^{p_j}$ with $p_1 + \cdots + p_r = p$, and thus $Y = [Y_1 \dots Y_r]$ with $\{Y_j \in F_{n\times p_j};\ 1 \le j \le r\}$, we infer that $\{W_{jj}' = \sum_{i=1}^{n} y_{ij}y_{ij}';\ 1 \le j \le r\}$, so that $\{\mathrm{tr}W_{jj} = \sum_{i=1}^{n} y_{ij}'\, y_{ij};\ 1 \le j \le r\}$ are quadratic forms in jointly Gaussian variables.

More general forms $\{\mathrm{tr}W_{jj}B_j;\ 1 \le j \le r\}$, with $\{B_j \in S_{p_j}^+;\ 1 \le j \le r\}$, are also quadratic forms in jointly Gaussian variables because $\{\mathrm{tr}W_{jj}B_j = \sum_{i=1}^{n} y_{ij}'\, B_j y_{ij};\ 1 \le j \le r\}$. The transformation $Y \to YB'$, with $B' = \mathrm{Diag}(B_1^{1/2}, \dots, B_r^{1/2})$, shows that YB' has the distribution $\mathscr{L}(YB') = N_{n\times p}(MB', I_n \times B\Sigma B')$, and thus $V = Z'Z$, with $Z = YB'$, has $\mathscr{L}(V) = W_p(n, B\Sigma B', B\Theta B')$. Partitioning $V = [V_{ij}]$ conformably with $W = [W_{ij}]$ shows that $\{\mathrm{tr}V_{jj} = \mathrm{tr}W_{jj}B_j;\ 1 \le j \le r\}$ are positive semidefinite quadratic forms in Gaussian variables, so that $\{B_1, \dots, B_r\}$ may be absorbed into parameters of the Wishart distribution. In particular, the case $n = 1$ yields the forms $[y_1'B_1y_1, \dots, y_r'B_ry_r]$ in $y = [y_1', \dots, y_r']'$ on $\mathbb{R}^p$ as before.

Representations in all these cases can be given in terms of standardized variables. Specifically, the quadratic forms $[y'A_1y, \dots, y'A_ry]$ in $y \in \mathbb{R}^p$, with $\mathscr{L}(y) = N_p(\mu, \Sigma)$, become $[z'B_1z, \dots, z'B_rz]$ with $\{B_j = \Sigma^{1/2}A_j\Sigma^{1/2};\ 1 \le j \le r\}$ and $z = \Sigma^{-1/2}y$ such that $\mathscr{L}(z) = N_p(\theta, I_p)$ with $\theta = \Sigma^{-1/2}\mu$. The further transformation $z \to P_j'z = u_j$, where P_j is an orthogonal matrix chosen to diagonalize $\Sigma^{1/2}A_j\Sigma^{1/2} \to P_j'\Sigma^{1/2}A_j\Sigma^{1/2}P_j = \Delta_j = \mathrm{Diag}(\delta_{j1}, \dots, \delta_{jp})$, yields equivalent forms

$$\{y'A_jy = z'B_jz = u_j'\,\Delta_j u_j;\ 1 \le j \le r\} \tag{2.1}$$

in the Gaussian vector $u = [u_1', \dots, u_r']'$ on $\mathbb{R}^{rp}$. Here $\mathscr{L}(u) = N_{rp}(\gamma, \Xi)$ with $\gamma = [\gamma_1', \dots, \gamma_r']'$ such that $\{\gamma_j = P_j'\Sigma^{-1/2}\mu;\ 1 \le j \le r\}$, and in partitioned form $\Xi = [P_i'P_j] \in S_{rp}^+$. Other cases may be treated similarly.

2.4 A Limit Rule. The following lemma is basic, where $\Phi_p(z)$ denotes the p-dimensional Gaussian *cdf* with moments to be specified, and Z^* denotes the random diagonal matrix $Z^* = \mathrm{Diag}(Z_1^{\alpha_1}, \dots, Z_p^{\alpha_p})$.

LEMMA 1. Let $\{Z_N;\ N = 1, 2, \dots\}$ be a sequence of random diagonal $(p\times p)$ matrices having nonzero means $E(Z_N) = M_N$ and the diagonal matrix Ξ_N^2 containing variances only, and let $Y_N = \Xi_N^{-1}(Z_N - M_N)$ be such that (i) $\lim_{N\to\infty} \mathscr{L}(Y_N) = \Phi(y)$ and (ii) $\lim_{N\to\infty} M_N^{-1}\Xi_N = 0$. If $W_N(\alpha) = \mathrm{Diag}(Z_1^{\alpha_1}, \dots, Z_p^{\alpha_p})$, then for every $\{\alpha_j \in (0,1];\ 1 \le j \le p\}$ we have $\lim_{N\to\infty} \mathscr{L}(W_N(\alpha)) = \Phi_p(w)$.

Proof. Let $\underset{\sim}{A} = \text{Diag}(\alpha_1, \ldots, \alpha_p)$; write $\underset{\sim}{Z}_N = \underset{\sim}{\Xi}_N \underset{\sim}{Y}_N + \underset{\sim}{M}_N$; and note on dropping subscripts that $\underset{\sim}{Z}^{\alpha} = \underset{\sim}{M}^{\alpha}(\underset{\sim}{I} + \underset{\sim}{M}^{-1}\underset{\sim}{\Xi}\underset{\sim}{Y})^{\alpha}$. Expanding each element of $(\underset{\sim}{I} + \underset{\sim}{M}^{-1}\underset{\sim}{\Xi}\underset{\sim}{Y})^{\alpha}$ in its binomial series, using hypothesis (ii) for N sufficiently large and the assumption that $\alpha_j \in (0, 1]$, gives

$$(\underset{\sim}{I} + \underset{\sim}{M}^{-1}\underset{\sim}{\Xi}\underset{\sim}{Y})^{\alpha} = \underset{\sim}{I} + \underset{\sim}{A}\underset{\sim}{M}^{-1}\underset{\sim}{\Xi}\underset{\sim}{Y} + o(\underset{\sim}{M}^{-1}\underset{\sim}{\Xi}\underset{\sim}{Y}). \tag{2.2}$$

On combining expressions and reinstating subscripts, we have

$$(\underset{\sim}{Z}_N - \underset{\sim}{M}_N)^{\alpha} = \underset{\sim}{A}\underset{\sim}{M}_N^{\alpha-1}\underset{\sim}{\Xi}_N\underset{\sim}{Y}_N + o(\underset{\sim}{M}_N^{\alpha-1}\underset{\sim}{\Xi}_N\underset{\sim}{Y}_N). \tag{2.3}$$

The lemma now follows from hypothesis (i) on taking limits in (2.3), where it is seen that $\underset{\sim}{Z}_N^{\alpha}$ *asymptotically is linear* in the limiting Gaussian matrix $\underset{\sim}{Y}_N$, and where an additional standardization may be used if needed to secure a proper limit for $\mathscr{L}(\underset{\sim}{W}_N(\underset{\sim}{\alpha}))$.

3. NORMALIZING TRANSFORMATIONS

We seek normalizing transformations $\{Q_j \to (Q_j)^{\alpha_j};\ 1 \le j \le r\}$ for standardized quadratic forms in jointly Gaussian variables. From Section 2.3 it suffices to consider $[U_1/\nu\theta_1, \ldots, U_r/\nu\theta_r]$, where $\{U_j = \text{tr}\underset{\sim}{W}_{jj};\ 1 \le j \le r\}$, such that $\mathscr{L}(\underset{\sim}{W}) = W_p(\nu, \underset{\sim}{\Sigma}, \underset{\sim}{\Theta})$ and $\{E(\text{tr}\underset{\sim}{W}_{jj}) = \nu\theta_j;\ 1 \le j \le r\}$. We first evaluate joint cumulants of $[U_1, \ldots, U_r]$ from their *cgf*; we next convert cumulants to moments; and we then expand typical moments of $\{V_j = (U_j/\nu\theta_j)^{\alpha_j};\ 1 \le j \le r\}$ to terms of required order in ν. We finally choose $\{\alpha_1, \ldots, \alpha_r\}$ so as to accelerate the convergence of certain moment sequences to those of jointly Gaussian variables. We outline these steps in the sections following, deferring further details to Appendix A.

3.1 The Joint Moments. Starting with the joint *chf* of $\underset{\sim}{W} = [\underset{\sim}{W}_{ij}]$ with argument $\underset{\sim}{T} \in S_p$ (*cf.* [2]) and expanding its *cgf* as in [11], we write the joint *cgf* of $[U_1, \ldots, U_r]$, with argument $\underset{\sim}{t} = [t_1, \ldots, t_r]'$, as

$$\psi_U(\underset{\sim}{t}) = \nu \sum_{s=1}^{\infty} i^s 2^{s-1} s^{-1} \text{tr}[\,(\underset{\sim}{T}_0\underset{\sim}{\Sigma})^{s-1}\underset{\sim}{T}_0(\underset{\sim}{\Sigma} + s\underset{\sim}{\Omega})] \tag{3.1}$$

where $\underset{\sim}{\Omega} = \nu^{-1}\underset{\sim}{\Theta}$ and $\underset{\sim}{T}_0 = \text{Diag}(t_1\underset{\sim}{I}_{p_1}, \ldots, t_r\underset{\sim}{I}_{p_r})$. The joint cumulant $\kappa_{s_1 \ldots s_r}$, of order $s = s_1 + \cdots + s_r$, is the coefficient of $i^s t_1^{s_1} \ldots t_r^{s_r}/s_1! \ldots s_r!$ in the power series (3.1). To simplify notation, partition $\underset{\sim}{W} = [\underset{\sim}{W}_{ij}]$, $\underset{\sim}{\Sigma} = [\underset{\sim}{\Sigma}_{ij}]$ and $\underset{\sim}{\Omega} = [\underset{\sim}{\Omega}_{ij}]$ conformably; let $\theta_j = \text{tr}(\underset{\sim}{\Sigma}_{jj} + \underset{\sim}{\Omega}_{jj})$ and $\omega_{js} = \text{tr}\underset{\sim}{\Sigma}_{jj}^{s-1}(\underset{\sim}{\Sigma}_{jj} + s\underset{\sim}{\Omega}_{jj})$ for $1 \le j \le r$; define

$$\sigma_s(i_1 i_2 i_2 i_3 \ldots i_s i_1) = \text{tr}\underset{\sim}{\Sigma}_{i_1 i_2}\underset{\sim}{\Sigma}_{i_2 i_3} \cdots \underset{\sim}{\Sigma}_{i_{s-1} i_s}(\underset{\sim}{\Sigma}_{i_s i_1} + s\underset{\sim}{\Omega}_{i_s i_1}) \tag{3.2}$$

where $s' = s - 1$ and $s = 1, 2, \ldots$; and write $\sigma_3(ijjiii) = \sigma_3(ij^2i^3)$, for example. The joint cumulants $\kappa_{s_1 \ldots s_r}$ of $[U_1, \ldots, U_r]$, of orders $1 \le s \le 4$, are summarized in Table 1 using the symmetry of $\underset{\sim}{\Sigma}$ and $\underset{\sim}{\Omega}$ as in Appendix A.

TABLE 1. Typical cumulants $\kappa_{s_1 \ldots s_r}$ of $[U_1, \ldots, U_r]$ of order s, $1 \le s \le 4$.

s	$t_1^{s_1} \ldots t_r^{s_r}$	Coefficient of $i^s t_1^{s_1} \ldots t_r^{s_r}/s_1! \ldots s_r!$
1	t_i	$\nu\theta_i$
2	t_i^2	$2\nu\omega_{i2}$
2	$t_i t_j$	$2\nu\sigma_2(ij^2i)$
3	t_i^3	$8\nu\omega_{i3}$
3	$t_i^2 t_j$	$8\nu[\sigma_3(ij^2i^3) + 2\sigma_3(i^3j^2i)]/3$
3	$t_i t_j t_k$	$8\nu[\sigma_3(ij^2k^2i) + \sigma_3(ik^2j^2i) + \sigma_3(ji^2k^2j)]/3$
4	t_i^4	$48\nu\omega_{i4}$
4	$t_i^3 t_j$	$24\nu[\sigma_4(i^5j^2i) + \sigma_4(i^3j^2i^3)]$
4	$t_i^2 t_j^2$	$8\nu[2\sigma_4(i^3j^4i) + 2\sigma_4(ij^2i^2j^2i) + \sigma_4(ij^4i^3) + \sigma_4(ji^4j^3)]$
4	$t_i^2 t_j t_k$	$8\nu[\sigma_4(i^3j^2k^2i) + \sigma_4(i^3k^2j^2i) + \sigma_4(ij^2i^2k^2i)$ $+ \sigma_4(ij^2k^2i^3) + \sigma_4(ji^4k^2j) + \sigma_4(ik^2i^2j^2i)]$

Moments of $[U_1, \ldots, U_r]$ are computed from Table 1 using known relations between moments and cumulants as in Appendix Table A1; see [6], for example. These computations give typical joint moments of orders $1 \le s \le 4$ as summarized in Table 2, and other moments follow similarly.

TABLE 2. Typical joint moments of $[U_1, U_2, U_3]$ of order s, $1 \le s \le 4$.

s	Symbol	Expression
1	μ'_{100}	$\nu\theta_1$
2	μ_{200}	$2\nu\omega_{12}$
2	μ_{110}	$2\nu\sigma_2(12^21)$
3	μ_{300}	$8\nu\omega_{13}$
3	μ_{210}	$8\nu[\sigma_3(12^21^3) + 2\sigma_3(1^32^21)]/3$
3	μ_{111}	$8\nu[\sigma_3(12^23^21) + \sigma_3(13^22^21) + \sigma_3(21^23^22)]/3$
4	μ_{400}	$48\nu\omega_{14} + 12\nu^2\omega_{12}^2$
4	μ_{310}	$24\nu[\sigma_4(1^52^21) + \sigma_4(1^32^21^3)] + 12\nu^2\omega_{12}\sigma_2(12^21)$
4	μ_{220}	$8\nu[2\sigma_4(1^32^41) + 2\sigma_4(12^21^22^21) + \sigma_4(12^41^3) + \sigma_4(21^42^3)]$ $+ 4\nu^2\omega_{12}\omega_{22} + 8\nu^2\sigma_2^2(12^21)$
4	μ_{211}	$\{8\nu[\sigma_4(1^32^23^21) + \sigma_4(1^33^22^21) + \sigma_4(12^21^23^21)]$ $+ \sigma_4(12^23^21^3) + \sigma_4(21^43^22) + \sigma_4(13^21^22^21)$ $+ 4\nu^2\omega_{12}\sigma_2(23^22) + 8\nu^2\sigma_2(12^21)\sigma_2(13^21)\}$

3.2 The Transformations. In what follows suppose that $\underset{\sim}{\Theta} = o(\nu)$ as $\nu \to \infty$, and conside 'he standardized variables $[U_1/\nu\theta_1, \ldots, U_r/\nu\theta_r]$ with $\{\nu\theta_j = E(U_j);\ 1 \le j \le r\}$ to ensure a proper limit. We establish in Theorem 1 that $[(U_1/\nu\theta_1)^{\alpha_1}, \ldots, (U_r/\nu\theta_r)^{\alpha_r}]$ are asymptotically Gaussian for every $\underset{\sim}{\alpha} \in D(r)$.

THEOREM 1. Suppose that $\mathscr{L}(\underset{\sim}{W}) = W_p(\nu, \underset{\sim}{\Sigma}, \underset{\sim}{\Theta})$ such that $\underset{\sim}{\Theta} = o(\nu)$ as $\nu \to \infty$. Then the variables $[U_1/\nu\theta_1, \ldots, U_r/\nu\theta_r]$, with $\{U_j = \mathrm{tr}\underset{\sim}{W}_{jj};\ 1 \le j \le r\}$, are asymptotically Gaussian as $\nu \to \infty$, as are $[(U_1/\nu\theta_1)^{\alpha_1}, \ldots, (U_r/\nu\theta_r)^{\alpha_r}]$ for every $\underset{\sim}{\alpha} \in D(r)$.

Proof. The variables $\underset{\sim}{W} = \underset{\sim}{Y}'\underset{\sim}{Y}$ when standardized, and thus $[U_1/\nu\theta_1, \ldots, U_r/\nu\theta_r]$ as marginals, are asymptotically Gaussian by central limit theory [11], verifying condition (i) of Lemma 1 for $\underset{\sim}{Z}_\nu = \mathrm{Diag}(U_1/\nu\theta_1, \ldots, U_r/\nu\theta_r)$. To verify condition (ii), identify $\underset{\sim}{M}_\nu$ and $\underset{\sim}{\Xi}_\nu$ as in Lemma 1 with $N = \nu$ and observe from Table 1 that

$$\lim_{\nu\to\infty} \underset{\sim}{M}_\nu^{-1}\underset{\sim}{\Xi}_\nu = \lim_{\nu\to\infty}(2/\nu)^{1/2}\mathrm{Diag}(\gamma_1, \ldots, \gamma_r) = \underset{\sim}{0} \tag{3.3}$$

where $\{\gamma_j = \omega_{j2}^{1/2}\theta_j^{-1};\ 1 \le j \le r\}$ and $\gamma_j = o(1)$ because $\underset{\sim}{\Theta} = o(\nu)$. Lemma 1 now asserts that the transformations $T(\underset{\sim}{\alpha}):\{U_j/\nu\theta_j \to (U_j/\nu\theta_j)^{\alpha_j};\ 1 \le j \le r\}$ yield asymptotically Gaussian variables for every $\{\alpha_j \in (0,1];\ 1 \le j \le r\}$, which completes our proof.

It remains to examine certain moment sequences of $[(U_1/\nu\theta_1)^{\alpha_1}, \ldots, (U_r/\nu\theta_r)^{\alpha_r}]$, which we now investigate. Letting $\{U_j = \nu\theta_j + e_j;\ 1 \le j \le r\}$ and $\{V_j = (U_j/\nu\theta_j)^{\alpha_j};\ 1 \le j \le r\}$, we evaluate the mixed moments

$$E(V_1^{s_1} \cdots V_r^{s_r}) = E\,[(1 + e_1/\nu\theta_1)^{\alpha_1 s_1} \cdots (1 + e_r/\nu\theta_r)^{\alpha_r s_r}] \tag{3.4}$$

on expanding each factor and finding expected values term-wise. For moments to third order, it suffices to consider the case $r = 3$ and to use $\{\alpha, \beta, \gamma\}$ in lieu of $\{\alpha_1, \alpha_2, \alpha_3\}$. The resulting expression is basic, namely,

$$E(V_1^r V_2^s V_3^t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{\nu}\right)^{i+j+k}\theta_1^{-i}\theta_2^{-j}\theta_3^{-k}a_i^r b_j^s c_k^t E(e_1^i e_2^j e_3^k) \tag{3.5}$$

where $\{a_i^r, b_j^s, c_k^t\}$ are binomial coefficients identified in Appendix A.

TABLE 3. Marginal moments of $V_1 = (\mathrm{tr}\underset{\sim}{W}_{11}/\nu\theta_1)^\alpha$ to terms of order $o(\nu^{-3})$.

Moment	Expression*
$\mu'_{100}(\alpha)$	$1 + \frac{1}{\nu\theta_1}(\alpha_{[1]}\phi_2) + \frac{\alpha_{[2]}}{\nu^2\theta_1^2}\left[\left(\frac{4}{3}\right)\phi_3 + \left(\frac{\alpha-3}{2}\right)\phi_2^2\right]$
$\mu_{200}(\alpha)$	$\frac{(2\alpha^2\phi_2)}{\nu\theta_1} + \frac{2\alpha^2(\alpha-1)}{\nu^2\theta_1^2}[4\phi_3 + (3\alpha-5)\phi_2^2]$
$\mu_{300}(\alpha)$	$\frac{4\alpha^3}{\nu^2\theta_1^2}[2\phi_3 + 3(\alpha-1)\phi_2^2]$

*$\alpha_{[t]} = \alpha(\alpha-1)\cdots(\alpha-t)$; $\phi_s = \mathrm{tr}\underset{\sim}{\Sigma}_{11}^{-1}(\underset{\sim}{\Sigma}_{11} + s\underset{\sim}{\Omega}_{11})/\mathrm{tr}(\underset{\sim}{\Sigma}_{11} + \underset{\sim}{\Omega}_{11})$.

To continue, identify moments about zero as $\mu'_{rst}(\alpha,\beta,\gamma) = E[V_1^r V_2^s V_3^t]$; let $\mu_{rst}(\alpha,\beta,\gamma)$ be the corresponding central moment; and let $\mu_{rs0}(\alpha,\beta)$, $\mu_{0st}(\beta,\gamma)$ and $\mu_{r00}(\alpha)$ be typical marginal moments. We next compute $\{\mu'_{rst}(\alpha,\beta,\gamma);\ r+s+t=1\}$ and $\{\mu_{rst}(\alpha,\beta,\gamma);\ 2 \le r+s+t \le 3\}$ to terms of order $o(v^{-3})$, applying expression (3.5) as often as needed to known relations between central and noncentral moments and using entries from Table 2. Moments of each one-dimensional marginal distribution are taken from [16] on replacing their θ_1 by our $v\theta_j$, for example. With $\alpha_{[t]} = \alpha(\alpha-1)\dots(\alpha-t)$ and $\{\phi_s = \omega_{1s}/v\theta_1;\ s=2,3,\dots\}$, these moments are given in Table 3 to terms of order $o(v^{-3})$ for $(U_1/v\theta_1)^\alpha$, with corresponding expressions for $(U_2/v\theta_2)^\beta$ and $(U_3/v\theta_3)^\gamma$.

TABLE 4. Nonvanishing terms to order $i+j \le 4$ and their coefficients in the expression (A.6) for the joint moment $\mu_{110}(\alpha,\beta)$ of $[V_1, V_2, V_3]$.

i	j	$(\frac{1}{v})^{i+j}$	$(\theta_1^i\theta_2^j)^{-1}$	$a_i^1 b_j^1$	$[E(e_1^i e_2^j) - E(e_1^i)E(e_2^j)]$
1	1	v^{-2}	$(\theta_1\theta_2)^{-1}$	$\alpha\beta$	$2v\sigma_2(12^21)$
2	1	v^{-3}	$(\theta_1^2\theta_2)^{-1}$	$\alpha(\alpha-1)\beta/2$	$8v[\sigma_3(12^21^3) + 2\sigma_3(1^32^21)]/3$
1	2	v^{-3}	$(\theta_1\theta_2^2)^{-1}$	$\alpha\beta(\beta-1)/2$	$8v[\sigma_3(21^22^3) + 2\sigma_3(2^31^22)]/3$
2	2	v^{-4}	$(\theta_1^2\theta_2^2)^{-1}$	$\alpha(\alpha-1)\beta(\beta-1)/4$	$8v[K_{220} + v\sigma_2^2(12^21)]$
3	1	v^{-4}	$(\theta_1^3\theta_2)^{-1}$	$\alpha(\alpha-1)(\alpha-2)\beta/6$	$12v[2K_{310} + v\omega_{12}\sigma_2(12^21)]$
1	3	v^{-4}	$(\theta_1\theta_2^3)^{-1}$	$\alpha\beta(\beta-1)(\beta-2)/6$	$12v[2K_{130} + v\omega_{22}\sigma_2(12^21)]$

TABLE 5. Nonvanishing terms to order $i+j \le 4$ and their coefficients in the expression (A.7) for the joint moment $\mu_{210}(\alpha,\beta)$ of $[V_1, V_2, V_3]$.

h	i	j	$(\frac{1}{v})^{h+i+j}$	$(\theta_1^{h+i}\theta_2^j)^{-1}$	Coefficient	$[E(e_1^h)E(e_1^i e_2^j) - E(e_1^h)E(e_1^i)E(e_2^j)]$
0	1	1	v^{-2}	$(\theta_1\theta_2)^{-1}$	0	$2v\sigma_2(12^21)$
0	2	1	v^{-3}	$(\theta_1^2\theta_2)^{-1}$	$\alpha^2\beta$	$8v[\sigma_3(12^21^3) + 2\sigma_3(1^32^21)]/3$
0	1	2	v^{-3}	$(\theta_1\theta_2^2)^{-1}$	0	$8v[\sigma_3(21^22^3) + 2\sigma_3(2^31^22)]/3$
0	2	2	v^{-4}	$(\theta_1^2\theta_2^2)^{-1}$	$\alpha^2\beta(\beta-1)/2$	$8v[K_{220} + v\sigma_2^2(12^21)]$
0	3	1	v^{-4}	$(\theta_1^3\theta_2)^{-1}$	$\beta\alpha^2(\alpha-1)$	$12v[2K_{310} + v\omega_{12}\sigma_2(12^21)]$
0	1	3	v^{-4}	$(\theta_1\theta_2^3)^{-1}$	0	$12v[2K_{130} + v\omega_{22}\sigma_2(12^21)]$
2	1	3	v^{-4}	$(\theta_1^3\theta_2)^{-1}$	$-\alpha^2(\alpha-1)\beta$	$4v^2\omega_{12}\sigma_2(12^21)$

We summarize in Tables 4, 5 and 6 the computations needed for $\mu_{110}(\alpha,\beta)$, $\mu_{210}(\alpha,\beta)$, and $\mu_{111}(\alpha,\beta,\gamma)$ up to terms of order $o(v^{-3})$, where expressions for $\{K_{ijk}\}$ are identified in Table 7. Details are given in Appendix A. The required partial sums for $\{\mu_{rst}(\alpha,\beta,\gamma)\}$ are found from each table on ignoring the first two columns, next multiplying the remaining expressions in each row, then summing these products over rows, and finally collecting terms in powers of v^{-1}.

TABLE 6. Nonvanishing terms to order $i+j+k \leq 4$ and their coefficients in the expression (A.8) for the joint moment $\mu_{111}(\alpha, \beta, \gamma)$ of $[V_1, V_2, V_3]$

$i\ j\ k$	$(\frac{1}{v})^{i+j+k}$	$(\theta_1^i \theta_2^j \theta_3^k)^{-1}$	$a_i^! b_j^! c_k^!$	$C(i,j,k)$*
1 1 1	v^{-3}	$(\theta_1\theta_2\theta_3)^{-1}$	$\alpha\beta\gamma$	$8v[\sigma_3(12^23^21) + \sigma_3(13^22^21) + \sigma_3(21^23^22)]/3$
2 1 1	v^{-4}	$(\theta_1^2\theta_2\theta_3)^{-1}$	$\alpha(\alpha-1)\beta\gamma/2$	$8v[K_{211} + v\sigma_2(12^21)\sigma_2(13^21)]$**
1 2 1	v^{-4}	$(\theta_1\theta_2^2\theta_3)^{-1}$	$\alpha\beta(\beta-1)\gamma/2$	$8v[K_{121} + v\sigma_2(21^22)\sigma_2(23^22)]$
1 1 2	v^{-4}	$(\theta_1\theta_2\theta_3^2)^{-1}$	$\alpha\beta\gamma(\gamma-1)/2$	$8v[K_{112} + v\sigma_2(31^23)\sigma_2(32^23)]$

*$C(i,j,k)$ is defined in expression (A.9).
**This expression derives from $\{8vK_{211} + 4v^2\omega_{12}\sigma_2(23^22) + 8v^2\sigma_2(12^21)\sigma_2(13^21) - 0 - (2v\omega_{12})[2v\sigma_2(23^22)] - 0 + 0\}$, for example.

TABLE 7. Identification of typical constants $\{K_{ijk}\}$.

i	j	k	K_{ijk}
3	1	0	$\sigma_4(1^52^21) + \sigma_4(1^32^21^3)$
2	2	0	$2\sigma_4(1^32^41) + 2\sigma_4(12^21^22^21) + \sigma_4(12^41^3) + \sigma_4(21^42^3)$
2	1	1	$[\sigma_4(1^32^23^21) + \sigma_4(1^33^22^21) + \sigma_4(12^21^23^21)$ $+ \sigma_4(12^23^21^3) + \sigma_4(21^43^22) + \sigma_4(13^21^22^21)]$

3.3 The Approximating Distribution. It remains to choose $\{\alpha, \beta, \gamma\}$. As all mixed central moments of $[V_1, V_2, V_3]$ of order 3 vanish up to terms of order $o(v^{-2})$ (compare Tables 5 and 6), we specifically choose $\{\alpha, \beta, \gamma\}$ so as to annihilate the leading term in expressions for each of $\mu_{300}(\alpha)$, $\mu_{030}(\beta)$, and $\mu_{003}(\gamma)$. The solution from Table 3 is $\alpha = 1 - 2\theta_1\omega_{13}/3\omega_{12}^2$, with similar expressions for β and γ. For determining $\{\alpha_1, \dots, \alpha_r\}$ in the general case, identical arguments yield

$$\alpha_j = 1 - 2\theta_j\omega_{j3}/3\omega_{j2}^2;\ 1 \leq j \leq r. \tag{3.6}$$

The foregoing developments support a Gaussian approximation to the joint distribution of $[V_1, \dots, V_r]$. Up to terms of order $o(v^{-2})$, the means $\underset{\sim}{\mu} = [\mu_1, \dots, \mu_r]'$ and dispersion parameters $\underset{\sim}{\Xi} = [\xi_{ij}]$ of the approximating distribution are given in the following theorem; expressions to order $o(v^{-3})$ are available from Table 3 and Table 4.

THEOREM 2. To terms of order $o(v^{-2})$, parameters of the Gaussian approximation to the joint distribution of $[V_1, \dots, V_r]$, with $\{V_j = (\mathrm{tr}\underset{\sim}{W}_{jj}/v\theta_j)^{\alpha_j};\ 1 \leq j \leq r\}$, are $\underset{\sim}{\mu} = [\mu_1, \dots, \mu_r]'$ and $\underset{\sim}{\Xi} = [\xi_{ij}]$, where

$$\mu_j = 1 + \omega_{j2}\alpha_j(\alpha_j - 1)/v\theta_j^2;\ 1 \leq j \leq r \tag{3.7}$$

$$\xi_{ij} = 2\alpha_i\alpha_j\sigma_2(ij^2i)/v\theta_i\theta_j;\ 1 \leq i,\ j \leq r \tag{3.8}$$

for every $\{\alpha_j \in (0, 1];\ 1 \le j \le r\}$.

In summary, the joint distribution of $[V_1, \dots, V_r]$ is approximately Gaussian as indicated. Equivalently, the variables

$$Z_j = \nu^{1/2}\theta_j[(U_j/\nu\theta_j)^{\alpha_j} - 1 - \omega_{j2}\alpha_j(\alpha_j - 1)/\nu\theta_j^2]/(2\omega_{j2}\,\alpha_j^2)^{1/2};\ 1 \le j \le r \tag{3.9}$$

may be taken to be approximately Gaussian having zero means, unit variances, and the correlation matrix $\underset{\sim}{R} = [\rho_{ij}]$, where

$$\rho_{ij} = \sigma_2(ij^2 i)/\omega_{i2}^{1/2}\omega_{j2}^{1/2},\ 1 \le i < j \le r \tag{3.10}$$

from $\rho_{ij} = \xi_{ij}/\xi_{ii}^{1/2}\xi_{jj}^{1/2}$ and (3.8). To terms of order $o(\nu^{-2})$, these correlations do not depend on the particular choice for $\{\alpha_1, \dots, \alpha_r\}$.

3.4 Some Special Cases. Special distributions merit further attention. As in Section 2.3, partition $\underset{\sim}{W} = [\underset{\sim}{W}_{ij}]$, $\underset{\sim}{\Sigma} = [\underset{\sim}{\Sigma}_{ij}]$, and $\underset{\sim}{\Theta} = \underset{\sim}{M}'\underset{\sim}{M} = [\underset{\sim}{\Theta}_{ij}]$ conformably; let $\{\lambda_j = \mathrm{tr}\underset{\sim}{\Theta}_{jj};\ 1 \le j \le r\}$; and suppose that $\{\underset{\sim}{\Sigma}_{jj} = \underset{\sim}{I}_{p_j};\ 1 \le j \le r\}$. Then the marginal distribution of $U_j = \mathrm{tr}\underset{\sim}{W}_{jj}$ is noncentral chi-squared having νp_j degrees of freedom and the noncentrality parameter λ_j; the corresponding central distribution of $[\mathrm{tr}\underset{\sim}{W}_{11}, \dots, \mathrm{tr}\underset{\sim}{W}_{rr}]$ has been studied in [10]. We determine from (3.6) that

$$\{\alpha_j = 1 - 2(p_j + \lambda_j)(p_j + 3\lambda_j)/3(p_j + 2\lambda_j)^2;\ 1 \le j \le r\}. \tag{3.11}$$

For central distributions having $\{\lambda_j = 0;\ 1 \le j \le r\}$, this reduces to $\{\alpha_j = 1/3;\ 1 \le j \le r\}$. The resulting transformation, $[U_1, \dots, U_r] \to [(U_1/\nu p_1)^{1/3}, \dots, (U_r/\nu p_r)^{1/3}]$, is a multivariate extension of the Wilson-Hilferty [32] transformation as applied to each variable. Up to terms of order $o(\nu^{-3})$, expressions for the means $[\mu_1, \dots, \mu_r]$ and the dispersion parameters $\underset{\sim}{\Xi} = [\xi_{ij}]$ of the approximating Gaussian distribution are given by

$$\mu_j = 1 - 2/9\nu p_j,\ \ 1 \le j \le r \tag{3.12}$$

$$\xi_{jj} = 2/9\nu p_j;\ 1 \le j \le r \tag{3.13}$$

$$\xi_{ij} = \frac{2\sigma_2(ij^2 i)}{9\nu p_i p_j} + \frac{8\sigma_2(ij^2 i)}{3^4\nu^2 p_i p_j}\left[\sigma_2(ij^2 i) - (1/2)\left(\frac{1}{p_i} + \frac{1}{p_j}\right)\right],\ \ i \ne j. \tag{3.14}$$

The corresponding correlation matrix $\underset{\sim}{R} = [\rho_{ij}]$ has elements

$$\rho_{ij} = \frac{\sigma_2(ij^2 i)}{p_i^{1/2}p_j^{1/2}} + \frac{4\sigma_2(ij^2 i)}{9\nu p_i^{1/2}p_j^{1/2}}\left[\sigma_2(ij^2 i) - (1/2)\left(\frac{1}{p_i} + \frac{1}{p_j}\right)\right];\ i \ne j. \tag{3.15}$$

Some numerical studies are reported in the following section based on the foregoing results.

4. NUMERICAL STUDIES

Our Gaussian approximations apply to various joint distributions as described in Section 2.3. Here we consider implementing these approximations, including a study of their accuracy in selected cases.

4.1 A Monotone Property. Although Gaussian distributions are perhaps best known among continuous multivariate distributions, even here available tables are limited in scope by the number of parameters required. The following facts are useful.

Our approximation can be implemented using available tables and a result of Slepian [31] which asserts that if $P_R(\cdot)$ and $P_\Gamma(\cdot)$ are Gaussian measures having zero means, unit variances, and correlation matrices $\underset{\sim}{R} = [\rho_{ij}]$ and $\underset{\sim}{\Gamma} = [\gamma_{ij}]$, respectively, such that $\{\gamma_{ij} \le \rho_{ij};\ 1 \le i \le j \le r\}$, then

$$P_R\{X_1 \le c_1, \dots, X_r \le c_r\} \ge P_\Gamma\{X_1 \le c_1, \dots, X_r \le c_r\} \tag{4.1}$$

for any scalars $\{c_1, \dots, c_r\}$. In the special case that $[Z_1, \dots, Z_r]$ are equicorrelated standard Gaussian variables having the correlation parameter ρ, the *cdf*

$$F_\rho(c, \dots, c) = P_\rho(Z_1 \le c, \dots, Z_r \le c) \tag{4.2}$$

has been tabulated in [7] for various values of r, ρ and c. If we now take $\rho^* = \min\{\rho_{ij};\ 1 \le i < j \le r\}$, Slepian's inequality assures that the Gaussian approximation for $[V_1, \dots, V_r]$ can be bounded below by $F_{\rho^*}(c_1, \dots, c_r)$. In the central case for which $\underset{\sim}{\Omega} = \underset{\sim}{0}$, we find that $\sigma_2(ij^2i) = \mathrm{tr}\underset{\sim}{\Sigma}_{ij}\underset{\sim}{\Sigma}_{ji} \ge 0$ and thus $\rho^* = 0$ can serve as a lower bound.

4.2 Accuracy of the Approximation. We compare approximate with exact probabilities using available tables or algorithms for the latter in selected cases. Two cases are of interset, namely, (i) the bivariate central chi-squared distributions of [20] using an algorithm developed in [15], and (ii) multivariate central chi-squared distributions studied in [22] and tabulated in [21]. In both studies the approximating Gaussian probabilities are taken directly from [7]. To avoid interpolating in these tables, we choose values of r, c and ρ given there, then refer to (3.9) to express the inequalities $\{Z_j \le c;\ 1 \le j \le r\}$ equivalently as $\{U_j \le c^*;\ 1 \le j \le r\}$ to find upper limits for $[U_1, \dots, U_r]$. Similarly we solve expression (3.15) with $\rho_{ij} = \rho$ in terms of ρ to find the parameters appropriate for the joint distribution of $[U_1, \dots, U_r]$. Exact probabilities were computed in the first case using sufficient terms of a series given in [15], and in the second case by interpolating in the tables of [21]. Details follow.

Let $\nu = 1, r = 2, p = 2n$, $\underset{\sim}{\Sigma}_{11} = \underset{\sim}{\Sigma}_{22} = \underset{\sim}{I}_n$, and $\underset{\sim}{\Sigma}_{12} = \tau \underset{\sim}{I}_n$. Then the joint *pdf* of $\{U_1 = \mathrm{tr}\, \underset{\sim}{W}_{11}, U_2 = \mathrm{tr}\, \underset{\sim}{W}_{22}\}$ is given in [20] as a series bilinear in the Laguerre polynomials. An algorithm provided in [15] was programmed and used to compute the probabilities listed in Appendix Table B1 for the cases $\{n = 1, 2, 3, 5, 15, 30 \text{ and } 50\}$. These are exact to the number of decimals reported. Four conclusions are suggested, other factors remaining constant. (i) For any n and τ the error of approximation tends to decrease as the included probability increases, *i.e.*, as c and c^* increase. In connection with statistical tests having upper-tail rejection regions, this indicates that the approximation is best where it is needed most. (ii) The error of approximation tends to decrease as n increases. This is indeed the rationale for our approximation, which is here seen to be asymptotic in n even when $\nu = 1$ in our earlier developments. (iii) The error of approximation tends to increase as τ and thus ρ increase. (iv) The Gaussian approximations appear tenable over most of the parameter values studied, especially in view of the fact that accuracy beyond the second decimal is seldom required in practice.

For the case $r = p$ and $\underset{\sim}{\Theta} = \underset{\sim}{0}$, the distribution of $[U_1, \ldots, U_r]$ reduces to that of the diagonal elements of a central Wishart matrix of order r having ν degrees of freedom as in [22]. For the special case that $\nu = 1$ and $\underset{\sim}{\Sigma} = [\sigma_{ij}]$ is a correlation matrix with equicorrelation parameter δ, extensive tables of this distribution are given in [21]. For $\{r = 3, 4, 6, 10\}$ and various values for the parameters ρ and c of the approximating Gaussian distributions, actual probabilities and their Gaussian approximations are given in Table B2 of Appendix B. Exact probabilities were obtained from [21] using a standard two-dimensional interpolation procedure (*cf.* [1], p. 882) to interpolate with respect to c^* and δ. Several conclusions are suggested by the data in Table B2 and are supported by other computations not reported here. Specifically, the error of approximation tends (i) to increase as either r or ρ increases, and (ii) to diminish towards upper tails of the distribution. When $p = r$ and $\nu = 1$, we accordingly recommend that use of the approximation be restricted to upper tails only. Despite the fact that our developments rest heavily on expressions asymptotic in ν, our approximations fare surprisingly well in upper tails of the distributions studied even when $\nu = 1$.

APPENDIX A

To supplement developments of Section 3, we consider first the joint cumulants and then moments of $[U_1, \ldots, U_r]$.

A.1 The Joint Cumulants. Suppose that $\mathscr{L}(\underset{\sim}{W}) = W_p(n, \underset{\sim}{\Sigma}, \underset{\sim}{\Theta})$. The joint *chf* of $\underset{\sim}{W}$ with symmetric argument $\underset{\sim}{T}$, as given in [2], yields the series expansion for its *cgf* as given in [11] in the form

$$\psi_W(\underset{\sim}{T}) = v\sum_{s=1}^{\infty} i^s 2^{s-1} s^{-1} \mathrm{tr}[\,(\underset{\sim}{T}\,\underset{\sim}{\Sigma})^{s-1}\underset{\sim}{T}(\underset{\sim}{\Sigma} + s\underset{\sim}{\Omega})] \tag{A.1}$$

where $\underset{\sim}{\Omega} = v^{-1}\underset{\sim}{\Theta}$. Standard arguments yield the joint marginal *cgf* of $[U_1, \ldots, U_r]$ on setting other arguments to zero in (A.1), giving (3.1). Letting $\underset{\sim}{G}(s) = (\underset{\sim}{T}_0\underset{\sim}{\Sigma})^{s-1}\underset{\sim}{T}_0(\underset{\sim}{\Sigma} + s\underset{\sim}{\Omega})$, we extract the joint cumulant $\kappa_{s_1 \ldots s_r}$ as the coefficient of $i^s t_1^{s_1} \ldots t_r^{s_r}/s_1! \ldots s_r!$ in the series

$$\psi_U(\underset{\sim}{t}) = v\sum_{s=1}^{\infty} i^s 2^{s-1} s^{-1}\mathrm{tr}\underset{\sim}{G}(s). \tag{A.2}$$

Proceeding recursively, with $\underset{\sim}{T}_0$ and $\underset{\sim}{\Sigma} = [\underset{\sim}{\Sigma}_{ij}]$ in partitioned form, we obtain

$$(\underset{\sim}{T}_0\underset{\sim}{\Sigma})^{s-1} = \left[\sum_{i_2=1}^{r} \cdots \sum_{i_{s'}=1}^{r} t_u \underset{\sim}{\Sigma}_{ui_2} t_{i_2}\underset{\sim}{\Sigma}_{i_2 i_3} \cdots t_{i_{s'}}\underset{\sim}{\Sigma}_{i_{s'}v}\right] \tag{A.3}$$

where $s' = s - 1$ and the expression inside brackets is the typical (u,v) block of $(\underset{\sim}{T}_0\underset{\sim}{\Sigma})^{s-1}$. Similarly the typical block of $\underset{\sim}{G}(s)$, with u and v ranging from 1 to r, is

$$\underset{\sim}{G}_{uv}(s) = \sum_{i_2=1}^{r} \cdots \sum_{i_s=1}^{r} t_u\underset{\sim}{\Sigma}_{ui_2} \cdots t_{i_{s'}}\underset{\sim}{\Sigma}_{i_{s'}i_s} t_{i_s}(\underset{\sim}{\Sigma}_{i_s v} + s\underset{\sim}{\Omega}_{i_s v}) \tag{A.4}$$

which, on summing $u = v$ from 1 to r, yields

$$\begin{aligned} tr\underset{\sim}{G}(s) &= \mathrm{tr}[\,(\underset{\sim}{T}_0\underset{\sim}{\Sigma})^{s-1}\underset{\sim}{T}_0(\underset{\sim}{\Sigma} + s\underset{\sim}{\Omega})] \\ &= \sum_{i_1=1}^{r} \cdots \sum_{i_s=1}^{r} t_{i_1} \cdots t_{i_s}\,\mathrm{tr}[\,\underset{\sim}{\Sigma}_{i_1 i_2} \cdots \underset{\sim}{\Sigma}_{i_{s'} i_s}(\underset{\sim}{\Sigma}_{i_s i_1} + s\underset{\sim}{\Omega}_{i_s i_1})]. \end{aligned} \tag{A.5}$$

Combining (A.5) and (A.2) yields the desired cumulants in terms of the parameters $\{\theta_j = \mathrm{tr}(\underset{\sim}{\Sigma}_{jj} + \underset{\sim}{\Omega}_{jj});\ 1 \le j \le r\}$, $\{\omega_{js};\ 1 \le j \le r,\ s = 1, 2, \ldots\}$, and $\{\sigma_s(i_1 i_2 i_2 i_3 \ldots i_s i_1);\ s = 1, 2, \ldots\}$ as defined in Section 3.1. These are summarized in Table 1 to include cumulants of order s, with $1 \le s \le 4$. Standard relations between moments and cumulants, as listed in Table A1, give values reported in Table 2.

TABLE A1. Relationships between typical central and noncentral moments and cumulants of order $s = s_1 + s_2 + s_3$, $1 \le s \le 4$.

s	Relationship
1	$\mu'_{100} = \kappa_{100}$
$2 \le s \le 3$	$\mu_{s_1 s_2 s_3} = \kappa_{s_1 s_2 s_3}$
4	$\mu_{400} = \kappa_{400} + 3\kappa_{200}^2$
4	$\mu_{310} = \kappa_{310} + 3\kappa_{200}\kappa_{110}$
4	$\mu_{220} = \kappa_{220} + \kappa_{200}\kappa_{020} + 2\kappa_{110}^2$
4	$\mu_{211} = \kappa_{211} + \kappa_{200}\kappa_{011} + 2\kappa_{110}\kappa_{101}$

A.2 Expansions for Moments of $[V_1, \ldots, V_r]$. Beginning with (3.4), we expand terms appearing there in their binomial series to obtain expression (3.5), where $\{a_i^r, b_j^s, c_k^t\}$ are binomial coefficients given by

$$a_i^r = r\alpha(r\alpha - 1) \ldots (r\alpha - i + 1)/i!$$
$$b_j^s = s\beta(s\beta - 1) \ldots (s\beta - j + 1)/j!$$
$$c_k^t = t\gamma(t\gamma - 1) \ldots (t\gamma - k + 1)/k!$$

For mixed moments of order two, repeated use of (3.5) in the expression $\mu_{110} = \mu'_{110} - \mu'_{100}\,\mu'_{010}$ gives

$$\mu_{110}(\alpha, \beta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{1}{v}\right)^{i+j} \theta_1^{-i} \theta_2^{-j} a_i^1 b_j^1 A(i,j) \tag{A.6}$$

where $A(i,j) = [E(e_1^i e_2^j) - E(e_1^i)E(e_2^j)]$. Similarly, from the identity $\mu_{210} = \mu'_{210} - 2\mu'_{110}\,\mu'_{100} + 2(\mu'_{100})^2\,\mu'_{010} - \mu'_{200}\,\mu'_{010}$ and (3.5), we determine that

$$\begin{aligned}\mu_{210}(\alpha, \beta) = &\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{1}{v}\right)^{i+j} \theta_1^{-i} \theta_2^{-j} (a_i^2 - 2a_i^1) b_j^1 B(0, i, j) \\ &- 2 \sum_{h=1}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{1}{v}\right)^{h+i+j} \theta_1^{-h-i} \theta_2^{-j} a_h^1 a_i^1 b_j^1 B(h, i, j)\end{aligned} \tag{A.7}$$

where $B(h, i, j) = E(e_1^h)[E(e_1^i e_2^j) - E(e_1^i)E(e_2^j)]$. Finally, the identity $\mu_{111} = \mu'_{111} - \mu'_{101}\,\mu'_{010} - \mu'_{011}\,\mu'_{100} - \mu'_{110}\,\mu'_{001} + 2\mu'_{100}\,\mu'_{010}\,\mu'_{001}$ after lengthy reduction yields

$$\mu_{111}(\alpha, \beta, \gamma) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{v}\right)^{i+j+k} \theta_1^{-i} \theta_2^{-j} \theta_3^{-k} a_i^1 b_j^1 c_k^1 C(i, j, k) \tag{A.8}$$

where

$$\begin{aligned}C(i, j, k) = [&E(e_1^i e_2^j e_3^k) - E(e_1^i e_3^k)E(e_2^j) - E(e_1^i)E(e_2^j e_3^k) \\ &- E(e_1^i e_2^j)E(e_3^k) + 2E(e_1^i)E(e_2^j)E(e_3^k)].\end{aligned} \tag{A.9}$$

Expansions (A.6), (A.7) and (A.8) may be truncated to approximate those moments to terms of specified order in v. It is required to compute a sufficient number of terms using expressions for $E(e_1^i e_2^j e_3^k)$ from Table 3 as central moments for $[U_1, U_2, U_3]$. Observe that $\{E(e_j) = 0;\ 1 \le j \le 3\}$; thus expressions for $\mu_{110}(\alpha, \beta)$, $\mu_{210}(\alpha, \beta)$ and $\mu_{111}(\alpha, \beta, \gamma)$ simplify somewhat. Specifically, $A(i,j) = 0 = B(0, i, j)$ if either $i = 0$ or $j = 0$; $B(1, i, j) = 0$ for all i and j; and $C(i, j, k) = 0$ if either $i = 0$, $j = 0$ or $k = 0$. Using these facts and properties of the binomial coefficients, we summarize in Tables 4, 5 and 6 the computations needed for $\mu_{110}(\alpha, \beta)$, $\mu_{210}(\alpha, \beta)$, and $\mu_{111}(\alpha, \beta, \gamma)$ up to terms of order $o(v^{-3})$, where the quantities $\{K_{ijk}\}$ are identified in Table 7.

APPENDIX B

TABLE B1. Exact probabilities for bivariate χ^2 distributions having n degrees of freedom. together with the Gaussian approximation* (A) having parameters c and ρ.

ρ	c	n = 1	2	3	5	15	30	50	A*
0.1	-2.0	.0000	.0001	.0004	.0005	.0007	.0008	.0008	.0009
0.1	-1.0	.0197	.0277	.0290	.0296	.0303	.0305	.0307	.0313
0.1	0.0	.2736	.2704	.2688	.2674	.2663	.2661	.2660	.2659
0.1	1.0	.7154	.7145	.7144	.7144	.7145	.7145	.7145	.7140
0.1	2.0	.9542	.9552	.9555	.9556	.9556	.9555	.9555	.9554
0.1	3.0	.9977	.9976	.9976	.9975	.9974	.9974	.9973	.9973
0.2	-2.0	.0000	.0002	.0004	.0007	.0010	.0011	.0012	.0014
0.2	-1.0	.0215	.0311	.0331	.0344	.0360	.0366	.0369	.0381
0.2	0.0	.2897	.2869	.2853	.2838	.2825	.2822	.2821	.2820
0.2	1.0	.7276	.7247	.7238	.7231	.7223	.7220	.7218	.7208
0.2	2.0	.9558	.9564	.9565	.9565	.9563	.9562	.9561	.9559
0.2	3.0	.9977	.9977	.9976	.9976	.9974	.9974	.9974	.9973
0.4	-2.0	.0000	.0002	.0007	.0012	.0020	.0023	.0024	.0029
0.4	-1.0	.0257	.0401	.0440	.0470	.0501	.0511	.0517	.0536
0.4	0.0	.3236	.3221	.3202	.3183	.3163	.3159	.3157	.3155
0.4	1.0	.7489	.7444	.7426	.7411	.7392	.7384	.7380	.7362
0.4	2.0	.9590	.9590	.9589	.9587	.9583	.9580	.9579	.9574
0.4	3.0	.9978	.9978	.9977	.9976	.9975	.9974	.9974	.9973
0.6	-2.0	.0000	.0003	.0011	.0023	.0040	.0045	.0048	.0055
0.6	-1.0	.0321	.0543	.0606	.0649	.0687	.0699	.0705	.0725
0.6	0.0	.3623	.3609	.3585	.3561	.3535	.3529	.3527	.3524
0.6	1.0	.7690	.7645	.7626	.7608	.7585	.7576	.7571	.7552
0.6	2.0	.9624	.9623	.9620	.9617	.9611	.9608	.9606	.9600
0.6	3.0	.9980	.9979	.9978	.9978	.9976	.9975	.9975	.9974
0.7	-2.0	.0000	.0004	.0016	.0034	.0056	.0063	.0066	.0074
0.7	-1.0	.0372	.0650	.0723	.0768	.0804	.0815	.0820	.0840
0.7	0.0	.3844	.3824	.3798	.3772	.3746	.3740	.3737	.3734
0.7	1.0	.7794	.7755	.7737	.7721	.7699	.7690	.7685	.7667
0.7	2.0	.9643	.9642	.9640	.9637	.9630	.9627	.9625	.9619
0.7	3.0	.9981	.9980	.9979	.9979	.9977	.9976	.9976	.9975

*Approximate values are $F\rho(c, c) = P_\rho(Z_1 \leq c,\ Z_2 \leq c)$ from [7].

TABLE B2. Exact (E) and approximate* (A) probabilities for an r-dimensional χ^2 distribution having $\nu = 1$ degrees of freedom.

	c							
	0.0		1.0		2.0		3.0	
ρ	E	A	E	A	E	A	E	A
				$r = 3$				
0.1	.1525	.1489	.6194	.6106	.9343	.9343	.9966	.9960
0.2	.1741	.1731	.6451	.6263	.9383	.9357	.9967	.9960
0.4	.2220	.2232	.6876	.6597	.9455	.9393	.9970	.9961
0.6	.2810	.2786	.7249	.6972	.9525	.9459	.9973	.9963
				$r = 4$				
0.1	.0868	.0871	.5413	.5259	.9158	.9140	.9955	.9946
0.2	.1077	.1130	.5794	.5506	.9226	.9166	.9957	.9947
0.4	.1581	.1692	.6406	.6007	.9341	.9233	.9962	.9948
0.6	.2267	.2334	.6922	.6539	.9447	.9338	.9967	.9953
				$r = 6$				
0.1	.0293	.0331	.4215	.3972	.8824	.8756	.9933	.9920
0.2	.0434	.0551	.4791	.4373	.8953	.8814	.9938	.9921
0.4	.0851	.1100	.5705	.5144	.9155	.8958	.9947	.9925
0.6	.1569	.1800	.6450	.5919	.9325	.9139	.9958	.9934
				$r = 10$				
0.1	.0037	.0066	.2683	.2404	.8258	.8065	.9893	.9867
0.2	.0079	.0184	.3469	.2983	.8515	.8202	.9903	.9870
0.4	.0277	.0605	.4780	.4072	.8876	.8507	.9924	.9881
0.6	.0822	.1281	.5853	.5146	.9154	.8838	.9943	.9902

*Approximate values are $F_\rho(c, \ldots, c) = P_\rho(Z_1 \le c, \ldots, Z_r \le c)$ from [7].

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Multidimensional Wilson-Hilferty [32] transformations support Gaussian approximations to certain joint distributions of quadratic forms in Gaussian variables. Central and noncentral distributions are studied and applications are noted. Parameters of the approximating distributions are given up to terms of order $o(v^{-3})$ in the degrees of freedom v. Numerical studies validate using these approximations over a range of parameters for an essential subclass of the distributions.